Exam. Code : 211003
Subject Code : 4966

M.Sc. Mathematics 3rd Semester (Batch 2020-22)

FUNCTIONAL ANALYSIS—I

Paper—MATH-571

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt **FIVE** questions in all, selecting at least **ONE** question from each section. The fifth question may be attempted from any section. All questions carry equal marks.

SECTION—A

1. (a) State and prove Holder's inequality. 8

(b) Let $C = C[0, 1]$ be the space of all continuous functions on $[0, 1]$ and define $\|f\| = \max|f(x)|$. Prove that C is a Banach space. 6

(c) Prove that a normed linear space X is complete if and only if every absolutely summable series is summable. 6

2. (a) Prove that L^{∞} is complete. 8

(b) Prove the Minkowski inequality for $0 < p < 1$. 6

SECTION—D

7. (a) If M is a closed linear subspace of a Hilbert space H, then $H = M \oplus M^{\perp}$. 5

(b) State and prove Schwarz's inequality. 5

(c) If P is a projection on a Banach space B and M and N are its range and null spaces, then M and N are closed linear subspaces of B such that $B = M \oplus N$. 5

(d) If M is a proper closed linear subspace of a Hilbert space H, then there exists a non zero vector y_0 in H such that $y_0 \perp M$. 5

8. (a) State and prove Bessel's inequality. 6

(b) Let H be a Hilbert space and f be an arbitrary functional in H*. Then there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H. 6

(c) Prove that l_2 is a Hilbert space. Show that the Parallelogram law is not true in l_1^n, $n > 1$. 4

(d) If B is a complex Banach space whose norm obeys the parallelogram law and if inner product is defined by $4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$, then B is a Hilbert space. 4

(c) If M is a closed linear subspace of a normed linear space N and the norm of a coset x + M in the quotient space N/M is defined by $\|x + M\| = \inf\{\|x + m\| : m \in M\}$. Then N/M is a normed linear space. Further if N is a Banach space, then so is N/M. 6

SECTION—B

3. (a) State and prove Hahn- Banach theorem. 8

(b) If M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M, then there exists a functional f_0 in N* such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$. 6

(c) Prove that $(l_1^n)^* = l_\infty^n$. 6

4. (a) State and prove Riesz theorem. 8

(b) If T is a continuous linear transformation of a normed linear space N into a normed linear space N' and if M is its null space. Show that T induces a natural linear transformation T' of N/M into N' such that $\|T'\| = \|T\|$. 6

(c) Prove that $\mathfrak{B}(N)$, the collection of all linear operators on N, is an algebra. 6

SECTION—C

5. (a) State and prove Uniform boundedness principle. 7

(b) If B and B' are Banach spaces and T is a continuous linear transformation of B onto B', then the image of each open sphere centered on the origin in B contains an open sphere centered on the origin in B'. 7

(c) Let T be an operator on a Banach space B, show that T has an inverse T^{-1} if and only if T* has an inverse $(T^*)^{-1}$ and in this case $(T^*)^{-1} = (T^{-1})^*$. 6

6. (a) If T is an operator on a normed linear space N, then its conjugate T* defined by $(T^*(f))(x) = f(T(x))$ is an operator on N* and the mapping $T \to T^*$ is an isometric isomorphism of $\mathfrak{B}(N)$ into $\mathfrak{B}(N^*)$, which reverses products and preserves the identity transformation. 8

(b) State and prove Open mapping theorem. 6

(c) If P is a projection on a Banach space B and M and N are its range and null spaces, then M and N are closed linear subspaces of B such that $B = M \oplus N$. 6